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LETTERS TO THE EDITOR

THE SLOW MOTION OF TWO TOUCHING FLUID SPHERES ALONG THEIR LINE OF CENTERS: ADDENDUM

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The interactions of fluid spheres are a subject of considerable current interest. In particular, several investigations (Wacholder & Weihs 1972; Haber *et al.* 1973; Reed & Morrison 1974) have recently examined the creeping motion of two fluid spheres along their line of centers. In each case, the effect of a neighbor on the force, F, exerted on a sphere is determined. This effect is expressed as a correction, β, to the force of Hadamard (1911) and Rybczynski (1911).

$$F = -6\pi\mu^e aU \frac{3\sigma + 2}{3\sigma + 3}\beta$$

where a is the sphere's radius and U is its velocity. μ^e is the viscosity of the external fluid and σ is the ratio of internal viscosity to external viscosity.

The effect of a touching neighbor on the motion of a fluid sphere was treated in Haber *et al.* (1973) and Reed & Morrison (1974). Haber *et al.* (1973), using the bispherical coordinate system, extended the analysis of Wacholder & Weihs (1972) which treats only separated spheres. For equal size spheres in contact, they obtained an integral expression for the correction to the particle drag. Employing the tangent sphere coordinate system, Reed & Morrison (1974) found an integral expression for the correction for spheres in contact and of arbitrary size.

The case of two spheres of vanishing internal viscosity corresponds closely to bubble motion through a liquid. In this case,

$$F = -4\pi\mu^e aU\beta$$

For equal size bubbles, the correction of Haber *et al.* is

$$\beta = 2\int_0^\infty \frac{e^{-x}\sinh^2 x(\cosh x + xe^x)}{\sinh^2 2x}\,dx$$

which they evaluate numerically, giving a value of 0.69309765 and claiming an accuracy of order 10^{-5}. Reed & Morrison present an equivalent expression

$$\beta = \frac{1}{2}\int_0^\infty \frac{e^{-\lambda}[\lambda\sinh\lambda + (\lambda + 1)\cosh\lambda]}{\cosh^2\lambda}\,d\lambda$$

and give a numerical value of 0.69315.

We note here that these integrals may be evaluated in closed form. The simplest procedure, perhaps, is to consider the integral in terms of the Laplace transform of the appropriate portion of the integrand. Use can then be made of the extensive tabulation of Laplace transform pairs.

The result of the integration is

$$\beta = \ln 2$$

so that the force on a bubble touching an equal bubble and moving along their line of centers is

$$F = -4\pi\mu^e aU \ln 2$$

ln 2 is approximately 0.69314718.

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